

Branes, Calabi–Yau Spaces, and Toroidal Compactification of the $N=1$ Six-Dimensional E_8 Theory

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We consider compactifications of the $N = 1$, $d = 6$, E_8 theory on tori to five, four, and three dimensions and learn about some properties of this theory. As a by-product we derive the $SL(2, \mathbb{Z})$ duality of the $N = 2$, $d = 4$, $SU(2)$ theory with $N_f = 4$. Using this theory on a D-brane probe we shed new light on the singularities of F-theory compactifications to eight dimensions. As another application we consider compactifications of F-theory, M-theory and the IIA string on (singular) Calabi-Yau spaces where our theory appears in spacetime. Our viewpoint leads to a new perspective on the nature of the singularities in the moduli space and their spacetime interpretations. In particular, we have a universal understanding of how the singularities in the classical moduli space of Calabi–Yau spaces are modified by worldsheet instantons to singularities in the moduli space of the corresponding conformal field theories.

October, 1996

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1. Introduction

Recent advances in local quantum field theories have uncovered a very rich structure. In particular, non-trivial fixed points in various dimensions have been found both in theories with four supersymmetries (for a review and earlier references see [1]) and with eight supersymmetries [2,3,4,5,6,7,8]. Also, it turns out to be interesting to study compactifications of a field theory as a function of the parameters of the compactification, thus interpolating between field theories in various dimensions.

One of the goals of this paper is to study the compactification on a torus of the simplest non-trivial fixed point with $N = 1$ supersymmetry in six dimensions (for a recent discussion of other fixed points in six dimensions see [9,10]). Its global symmetry is E_8 and it was first found in the study of small E_8 instantons in string theory [11,12]. The compactified theory depends on various parameters: the moduli of the torus and twists in the boundary conditions which break E_8 to its subgroups.

Our analysis proceeds in parallel from three different points of view:

1. The $d = 6$ E_8 theory is still mysterious. Although it looks like a local quantum field theory, it does not have a Lagrangian description. We expect that by studying the properties of this theory a simple presentation of the theory which makes its behavior manifest will emerge. Hopefully, such a presentation will also be useful in other field theories.
2. The $d = 6$ E_8 theory is the low energy description of five-branes in M-theory near the “end-of-the-world” brane. When this eleven-dimensional theory and the five-brane are compactified on a torus, the low energy theory on the brane is the theory mentioned above. Using dualities, this is the theory on lower dimension probes in compactifications of the type I' on $\mathbf{S}^1/\mathbb{Z}_2$ [4], F-theory [13] on K3 [14] and M-theory on K3 [15]. The compactified $d = 6$ E_8 theory thus tells us about the behavior of these compactifications.
3. The $d = 6$ E_8 theory is the low energy theory of the heterotic theory compactified on K3 near the limit as an E_8 instanton in the compactification shrinks to zero size. Another description of this theory is the compactification of F-theory on a singular Calabi–Yau space [16,12,17,18]. After compactification on another circle, this becomes the low energy theory of M-theory compactified on the same Calabi–Yau space. Varying parameters to break E_8 to its subgroups will alter the singularity type of the Calabi–Yau space. After further compactification on a circle we find a compactification of the type IIA theory on the same singular Calabi–Yau space. Therefore, an

understanding of this field theory can be achieved by relating it to the singularities of the Calabi–Yau compactifications and the way they are corrected by worldsheet instanton effects. Note that this field theoretic understanding of these compactifications makes it obvious that the nature of the singularities in the conformal field theory moduli space is independent of the details of the underlying Calabi–Yau space; they depend only on the singularities of that space.

Although these three applications are distinct, we will often find it easier to make an argument based on one point of view than on the others. Then we can translate the conclusion to learn about the other applications.

The six-dimensional theory has no free parameters. The moduli space of vacua has two branches. The Coulomb branch is \mathbf{R}^+ . The real scalar field which parameterizes it is in a tensor multiplet. There is also a Higgs branch isomorphic to the moduli space of E_8 instantons.

In lower dimensions new parameters, which are associated with the compactification appear. In five dimensions one real parameter is R_6 —the radius of the circle. We can also couple the E_8 currents of the global symmetry to background gauge fields. This leads to eight more real parameters (the Wilson lines around the circle). It is convenient to think of them as background superfields [19]. In this case, all of them are in vector multiplets [20] of the five-dimensional theory. Specifically, $\frac{1}{R_6}$ is the scalar of a background vector superfield.

In four dimensions we get more parameters. First, the eight scalars in the E_8 background gauge fields become complex. Second, the compactified \mathbf{T}^2 leads to two other vector superfields. Three of the four real scalars in these superfields are the two sides of the torus R_5 and R_6 , and the angle between them φ . The fourth one, which is needed for supersymmetry, is a background \tilde{B}_{56} field. $\tilde{B}_{\mu\nu}$ is a background two-form whose three-form field strength is self-dual. It is a component of the gravity multiplet in six dimensions. More explicitly,

$$\begin{aligned} v_5 &= \frac{1}{2\sqrt{2}\pi R_5 \sin \varphi} e^{-i\varphi/2 + i\tilde{B}/2} \\ v_6 &= \frac{1}{2\sqrt{2}\pi R_6 \sin \varphi} e^{i\varphi/2 + i\tilde{B}/2} \end{aligned} \tag{1.1}$$

are the scalar components of background vector superfields.

Since all these parameters are components of vector superfields, the way they affect the solutions is very restricted. First, the Higgs branch is independent of their values.

The only dependence follows from symmetry breaking—when the eight parameters break E_8 to a subgroup G , the Higgs branch is the moduli space of G instantons (when G is a product of several simple factors, there are several distinct Higgs branches). Second, in five dimensions the coupling of these vectors is very limited and essentially determined. In four dimensions there is more freedom than in five dimensions. But the constraints of supersymmetry are still useful—they imply that the elliptic curve describing the Coulomb branch [21] varies holomorphically in these parameters.

In section 2 we consider the compactification to five dimensions. We will see how the theories discussed in [4,5,22] are obtained. Also, we will show that there is always one more singularity in the moduli space and we will discuss its interpretation. In terms of the six dimensional theory it arises from a string with winding number $L_6 = 1$ and momentum $P_6 = 1$ which becomes massless at that point. We will discuss this singularity both from the point of view of the theory on the brane and from the point of view of M-theory compactification.

In section 3 we initiate a study of compactification to four dimensions following [23]. We first restrict ourselves to the subspace of the parameter space preserving the E_8 symmetry and study its properties. We then break the E_8 symmetry with Wilson lines and discuss more general situations. For special values of the Wilson lines we recover the $N = 2$, $d = 4$, $SU(2)$ theory with $N_f = 4$. Our six-dimensional viewpoint leads to a derivation of its $SL(2, \mathbb{Z})$ duality [24] as a manifestation of the $SL(2, \mathbb{Z})$ which acts on the torus we compactify on.

In section 4 we study the singularities of the four-dimensional theory in more detail. The possible singularities were classified by Kodaira [25]. We identify each of them with a four-dimensional field theory. Then, we consider the compactification of the various five-dimensional field theories on a circle and map them to singularities in four dimensions.

In section 5 we use the results of section 4 in the context of the theory on a three-brane in F-theory compactification to eight dimensions and in the context of type IIA compactifications to four dimensions. In particular, we shed new light on the behavior of the moduli space of the conformal field theories on the string worldsheet as corrected by worldsheet instantons near singularities of the geometry. Some of these singularities are understood as a result of non-trivial dynamics in a four-dimensional field theory. The short distance degrees of freedom of this field theory are visible in five dimensions.

In section 6 we make some comments about the compactification to three dimensions. Finally, in the appendices we give some more technical details.

2. From six to five dimensions

We start with the six-dimensional $N = 1$ supersymmetric field theory associated with small E_8 instantons. It has E_8 global symmetry. Being at a fixed point of the renormalization group it is exactly scale invariant. The Higgs branch of the theory is isomorphic to the moduli space of E_8 instantons. It is a hyper-Kähler manifold with E_7 symmetry and E_8 action. The massless hypermultiplets transform as $\frac{1}{2}(\mathbf{56}) + \mathbf{1}$ under E_7 where the $\frac{1}{2}$ stands for half hypermultiplets.

The Coulomb branch of the theory is \mathbf{R}^+ . The low energy degrees of freedom along this Coulomb branch are in a tensor multiplet which includes a two-form with self-dual field strength, a scalar Φ and a fermion. The kinetic term for Φ determines the metric on the Coulomb branch

$$\mathcal{L}_6 = \frac{1}{32\pi}(\partial\Phi)^2. \quad (2.1)$$

Note that Φ has dimension two and this effective Lagrangian (like the full theory) is scale invariant. More precisely, scale invariance is spontaneously broken along the Coulomb branch. The expectation value of Φ determines the tension of BPS strings

$$T = \sqrt{2}Z = \Phi. \quad (2.2)$$

Arguments based on string theory show that such BPS strings exist and they carry a chiral E_8 current algebra.

Since parameters are introduced in these theories by coupling them to background vector superfields, and since those do not have scalars, these quantum field theories do not have relevant operators which preserve the super Poincaré symmetry.

Let us compactify this theory on \mathbf{S}^1 of radius R_6 to five dimensions. For large $\Phi \gg \frac{1}{R_6^2}$ the massive modes decouple from the light modes. Therefore, in this regime the five-dimensional theory is obtained by dimensional reduction of (2.1)

$$\mathcal{L}_5 = \frac{1}{16}R_6(\partial\Phi)^2 \quad \text{for } \Phi \gg \frac{1}{R_6^2}. \quad (2.3)$$

The natural parameter on the Coulomb branch in five dimensions has dimension one. It is

$$\phi = \sqrt{2\pi}R_6\Phi \quad (2.4)$$

and (2.3) becomes

$$\mathcal{L}_5 = \frac{1}{32\pi^2 R_6}(\partial\phi)^2 \quad \text{for } \phi \gg \frac{1}{R_6}. \quad (2.5)$$

Similarly we can reduce the rest of the Lagrangian. The two-form becomes a gauge field with a ϕ -independent coupling constant

$$t(\phi) = \frac{16\pi^2\sqrt{2}}{g_{eff}^2(\phi)} = \frac{\sqrt{2}}{R_6} \quad \text{for } \phi \gg \frac{1}{R_6}. \quad (2.6)$$

Our normalization is such that $\frac{8\pi}{g_{eff}^2(\phi)} = \frac{\partial\phi_D}{\partial\phi}$ and $\phi_D \approx \Phi/\sqrt{2}$, $\phi \approx \sqrt{2}\pi R_6 \Phi$ for $\phi \gg \frac{1}{R_6}$. This definition of g_{eff} differs by a multiplicative factor from the one used in [4]; $t(\phi)$ is as in [5].

At small ϕ the five-dimensional theory has been analyzed [4] with the result

$$t(\phi) = 2\phi \quad \text{for } 0 \leq \phi \ll \frac{1}{R_6}. \quad (2.7)$$

Note that this behavior is consistent with the scale invariance of the five-dimensional theory at the origin at long distance. The two asymptotics (2.6) and (2.7) must be sewed in a way consistent with the restrictions from supersymmetry. The slope $\frac{\partial t(\phi)}{\partial\phi}$ can have discontinuities which are a positive integer multiple of -2 . Therefore, the only solution is

$$t(\phi) = \begin{cases} 2\phi & \text{for } 0 < \phi < \frac{1}{\sqrt{2}R_6} \\ \frac{\sqrt{2}}{R_6} & \text{for } \frac{1}{\sqrt{2}R_6} < \phi \end{cases} \quad (2.8)$$

That transition is exactly the one associated with a single hypermultiplet with electric charge one whose mass is $\sqrt{2}|\phi - \frac{1}{\sqrt{2}R_6}|$. The shift by $\frac{1}{R_6}$ in the mass indicates that the central charge for particles is

$$Z = n_e\phi - \frac{P_6}{\sqrt{2}R_6} \quad (2.9)$$

where n_e is the electric charge and P_6 is a charge of a global symmetry. The R_6 dependence of Z suggests that it can be interpreted as momentum in the compact direction. The particle which leads to the transition has $n_e = P_6 = 1$ and for consistency we have to assume that there are no other particles which become massless on the Coulomb branch away from $\phi = 0$.

Better understanding of the theory in six dimensions that we started with will have to explain why it has exactly one such particle with these quantum numbers. In the meantime, we note that since the electric charge n_e arises as the winding number of the six-dimensional string around the circle, our state has $L_6 = P_6 = 1$. States with $L_6 = P_6 = n$ must be interpreted as multiple states rather than as other elementary states in five dimensions.

Since our state behaves as an ordinary particle in five dimensions, it can have arbitrary five-momentum. Hence we can summarize the restriction on the quantum numbers by

$$\vec{L} \cdot \vec{P} = 1. \quad (2.10)$$

This is the equation which will have to be explained by better understanding of the six-dimensional theory.

Using (2.8) it is easy to find

$$\phi_D(\phi) = \begin{cases} \frac{\phi^2}{2\sqrt{2}\pi} & \text{for } 0 < \phi < \frac{1}{\sqrt{2}R_6} \\ \frac{\phi}{2\pi R_6} - \frac{1}{4\sqrt{2}\pi R_6^2} & \text{for } \frac{1}{\sqrt{2}R_6} < \phi \end{cases} \quad (2.11)$$

where the integration constant was set such that ϕ_D is continuous. We see that for $\frac{1}{\sqrt{2}R_6} < \phi$ strings have the tension $(\Phi - \frac{1}{4\pi R_6^2})$ —it is as in the six-dimensional theory up to a finite shift which vanishes as $R_6 \rightarrow \infty$. For $0 < \phi < \frac{1}{\sqrt{2}R_6}$ the tension of strings is $\phi^2/2\pi$ and they become tensionless at $\phi = 0$.

Equation (2.8) has a natural interpretation as the coupling constant in the effective theory on a D4-brane [26] in string theory. It describes the interaction of a D4-brane probe with a background orientifold at $\phi = 0$ where the coupling diverges [4]. Since the effective coupling for $\phi > \frac{1}{\sqrt{2}R_6}$ is constant, we can add arbitrarily far away, around $\phi = \phi_0 \gg \frac{1}{\sqrt{2}R_6}$, eight more background D8-branes and an orientifold to have a compactification of the type IIA theory on an orientifold $\mathbf{S}^1/\mathbb{Z}_2$. This theory is equivalent to M-theory compactified on $\mathbf{S}^1 \times (\mathbf{S}^1/\mathbb{Z}_2)$ where the two radii are R_6 and ϕ_0/π . In the limit of large ϕ_0 our theory focuses on the vicinity of one end-of-the-world brane in M-theory compactified on \mathbf{S}^1 of radius R_6 . The effective theory on a five-brane of M-theory wrapped on the \mathbf{S}^1 is the theory described above.

This analysis clarifies the behavior of the corresponding tensionless string in F-theory. Suppose we have a six-dimensional F-theory model with an almost-tensionless string which in the limit of zero tension is described by the E_8 theory. Concretely, this means that the base of the elliptic fibration contains a two-sphere Σ of self-intersection -1 whose area goes to zero in the limit. When this F-theory model is compactified on a circle, the resulting five-dimensional theory is dual to M-theory on the total space of the elliptic fibration. The almost-tensionless string in six dimensions gives rise to both a particle and a string in five dimensions, depending on whether we wrap it on the circle or not. From the M-theory perspective, the string comes from wrapping the five-brane around the four-manifold S

which lies over Σ , and the particle comes from wrapping the two-brane around a two-manifold (isomorphic to Σ) within the four-manifold.

As shown in [17] and further discussed in [27], in the M-theory model with any nonzero value of $1/R_6$, as the area of Σ goes to zero, the volume of S remains positive. The massless particle at the zero-area limit is the signal of a flop transition, and indeed the total space can be flopped. Continuing further in the moduli space, the volume of S can then be shrunk to zero. In other words, *the particle becomes massless at a different parameter value from where the string becomes tensionless*. This second transition is described by the five-dimensional E_8 theory of [4]; since it is an interacting theory, attempts to describe it in terms of free fields are likely to lead to confusing results such as an infinite number of light states. It is crucial to stress that there is nothing pathological about the low energy theory.

By adding Wilson lines with values in E_8 , the E_8 symmetry can be broken. Then, various different critical theories can be found [4]. At $\phi \neq 0$ the only special theories are those of n massless hypermultiplets. We denote them by A_{n-1} . Their symmetry is $SU(n)$. At $\phi = 0$ we can have an $SU(2)$ theory with n hypermultiplets which we denote by D_n . Its symmetry is $SO(2n)$. For $n = 0$ there is a possibility of turning on a discrete θ parameter [22] and therefore there are two theories D_0 and \tilde{D}_0 differing only in their massive spectra [5]. If the coupling constant diverges at $\phi = 0$ we also find a series of E_n theories with $n = 0, \dots, 8$ with symmetry E_n : E_8 , E_7 , E_6 , $E_5 = Spin(10)$, $E_4 = SU(5)$, $E_3 = SU(3) \times SU(2)$, $E_2 = SU(2) \times U(1)$ and $E_1 = SU(2)$ (E_0 has no symmetry). The final possibility is the theory \tilde{E}_1 whose symmetry is $U(1)$.

As we said above, these theories are obtained in string compactifications to nine dimensions. The A_n theories are obtained on a D4-brane probe near $n + 1$ coalescing D8-branes. The D_n theories are obtained near an orientifold with n D8-branes. The E_n theories are obtained when the string coupling diverges at the orientifold.

The other application of these theories is the low energy limit of M-theory compactified on a Calabi–Yau space X to five dimensions. They correspond to the various singularities of Calabi–Yau spaces [18,5,22]. The A_n theories are obtained from $n+1$ rational curves in a single homology class shrinking to a point. The D_n theories correspond to a rational ruled surface S with n singular fibers, shrinking to a rational curve. In the generic situation, the components of the singular fibers will all lie in a common homology class γ , and the fiber itself will be in class 2γ . Finally, the E_n cases correspond to a del Pezzo surface S shrinking to a point. In the generic situation, the image of the restriction map $H^{1,1}(X) \rightarrow H^{1,1}(S)$ will be a one-dimensional space, spanned by $c_1(S)$.

3. First look at compactification to four dimensions

3.1. Preliminaries

In this section we begin an analysis of the compactification of the E_8 theory to four dimensions on a two torus. The generic low-energy fields comprise a $U(1)$ vector multiplet of $N = 2$ in four dimensions. The Lagrangian can be constructed [23] from an elliptic curve as in [21].

The \mathbf{T}^2 torus has sides R_5, R_6 and angle φ . We define, as in (1.1),

$$\begin{aligned} v_5 &= \frac{1}{2\sqrt{2}\pi R_5 \sin \varphi} e^{-i\varphi/2 + i\tilde{B}/2} \\ v_6 &= \frac{1}{2\sqrt{2}\pi R_6 \sin \varphi} e^{i\varphi/2 + i\tilde{B}/2}. \end{aligned} \quad (3.1)$$

Its complex structure is

$$\begin{aligned} \sigma &= \frac{R_5}{R_6} e^{i\varphi} = \frac{v_6}{v_5} \\ q &= e^{2\pi i \sigma}. \end{aligned} \quad (3.2)$$

We first consider the region far along the flat directions. In that region we can neglect all effects associated with massive modes and simply dimensionally reduce the six dimensional free tensor multiplet. The bosonic fields of the tensor multiplet are Φ and $B_{\mu\nu}^{(-)}$. There is no Lagrangian for the anti-self-dual two-form but we can start with

$$L_{6D} = \frac{1}{32\pi} (d\Phi)^2 + \frac{1}{32\pi} (dB)^2 \quad (3.3)$$

and set the self-dual part of B to zero. The coupling constant for B is determined by anti-self-duality. In 4D we find the fields

$$\phi_1 = 2\pi R_6 \Phi, \quad A_\mu = B_{\mu 6}, \quad \phi_2 = \frac{1}{2\pi R_5} B_{56}. \quad (3.4)$$

In terms of the field-strength $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ we have

$$\partial_{[\mu} B_{\nu]5} = \frac{R_5 \sin \varphi}{R_6} \tilde{F}_{\mu\nu} + \frac{R_5 \cos \varphi}{R_6} F_{\mu\nu} = \text{Im} \left[\sigma (\tilde{F}_{\mu\nu} + i F_{\mu\nu}) \right]. \quad (3.5)$$

The 4D Lagrangian is thus

$$L_{4D} = \frac{\text{Im } \sigma}{32\pi} (d\phi_1)^2 + \frac{\text{Im } \sigma}{32\pi \sin^2 \varphi} (d\phi_2)^2 + \frac{\text{Im } \sigma}{32\pi} F^2 + \frac{\text{Re } \sigma}{32\pi} F \tilde{F}. \quad (3.6)$$

We define

$$a = \eta \frac{1}{\sqrt{2}} \left(\phi_1 + \frac{i}{\sin \varphi} \phi_2 \right), \quad (3.7)$$

$$a_D = \sigma a,$$

where η is an arbitrary phase which we will set below. In terms of a and a_D the Lagrangian reads:

$$\frac{1}{32\pi} \text{Im} \left[(da)(d\bar{a}_D) + \left(\frac{da_D}{da} \right) (F^2 + iF\tilde{F}) \right]. \quad (3.8)$$

The coordinate a lives on a cylinder since

$$\phi_2 \sim \phi_2 + \frac{1}{R_5}. \quad (3.9)$$

This identification follows from large gauge transformations in five dimensions which wind around the circle whose radius is R_5 . In five and six dimensions the moduli space is $\mathbf{R}^+ = \mathbf{R}/\mathbb{Z}_2$. Therefore, a good global coordinate on the moduli space is [28]

$$u = 2 \cosh(2\pi R_5 \sin \varphi \phi_1 + 2\pi i R_5 \phi_2). \quad (3.10)$$

so we find

$$a = v_5 \cosh^{-1}(u/2), \quad (3.11)$$

$$a_D = v_6 \cosh^{-1}(u/2).$$

In terms of the six dimensional variables Φ and B_{56}

$$u = 2 \cosh(4\pi^2 R_5 R_6 \sin \varphi \Phi + i B_{56}). \quad (3.12)$$

This agrees with the fact that the masses of winding states of the string around R_6 and R_5 should be a and a_D on one hand and on the other hand $2\pi R_6 \Phi$ and $2\pi R_5 \Phi$ (for $B_{56} = 0$) since Φ is the tension of the string. Furthermore, for large Φ we can interpret $\frac{1}{u} \approx e^{-4\pi^2 R_5 R_6 \sin \varphi \Phi - i B_{56}}$ as an instanton factor. The relevant instanton is a string which wraps our torus.

The central charge formula can be derived by matching to our five dimensional expressions and by using the symmetry $5 \leftrightarrow 6$

$$Z = n_e a + n_m a_D - 2\pi i (P_5 v_5 - P_6 v_6). \quad (3.13)$$

Here P_5 and P_6 are the momenta around R_5 and R_6 measured in integer units.

The monodromy of (3.11) around $u = \infty$ is

$$\begin{aligned} a &\approx v_5 \log u \longrightarrow a + 2\pi i v_5, \\ a_D &\approx v_6 \log u \longrightarrow a_D + 2\pi i v_6. \end{aligned} \tag{3.14}$$

This agrees with (3.13) if we change

$$\begin{aligned} n_e &\longrightarrow n_e, \\ n_m &\longrightarrow n_m, \\ P_5 &\longrightarrow P_5 - n_e, \\ P_6 &\longrightarrow P_6 + n_m, \end{aligned} \tag{3.15}$$

This non-trivial monodromy becomes trivial when we consider the action on $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$. Therefore, it is trivial when reduced to $SL(2, \mathbb{Z})$ —it does not act on $\tau = \frac{\partial a_D}{\partial a}$. Such non-trivial monodromies which include $SL(2, \mathbb{Z})$ as a quotient were observed in [24]. We see that although the gauge charges n_e and n_m transform trivially, the global charges P_5 and P_6 mix with the gauge charges.

We have seen in the previous section that there is one additional singularity at $\phi = \frac{1}{\sqrt{2}R_6}$ in 5D. Similarly, in 4D there should thus be two additional singularities—one at $a = \frac{1}{\sqrt{2}R_6}$ and one at $a_D = \frac{1}{\sqrt{2}R_5}$. The $SL(2, \mathbb{Z})$ monodromy around the first one is $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and around the second one is $S^{-1}TS = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Since the $SL(2, \mathbb{Z})$ monodromy around $u = \infty$ is trivial, we learn that the monodromy around the remaining singularities is $(S^{-1}TST)^{-1} = ST$ which is indeed what we expect from an E_8 singularity.

In five dimensions we identified $n_e = L_6$. It is clear that in four dimensions we should identify $n_m = L_5$. The monodromy (3.15) changes the quantum numbers of the particles at these singularities. Therefore, if a particle with (P_6, L_6, P_5, L_5) exists, there should also be a particle with $(P_6 - L_5, L_6, P_5 + L_6, L_5)$ (note that equation (2.10) is invariant under such a change in the charges). Therefore, the particle at one singularity has $(P_6 = 1, L_6 = 1, P_5 = n, L_5 = 0)$ for arbitrary n . All these particles should exist and are permuted under the monodromy (3.15). This fact has a simple interpretation. Upon reduction to five dimensions we found a particle with $p_6 = L_6 = 1$. As a five-dimensional particle it has arbitrary momentum. In particular, its P_5 must be arbitrary. A similar discussion applies to the particle at the other singularity after a substitution $5 \leftrightarrow 6$ in all these equations.

3.2. The elliptic curve

To determine the elliptic curve we pick an auxiliary parameter u on the moduli space. We assume that large u corresponds to large tension of the strings.

The elliptic curve

$$y^2 = x^3 - f(u, w_1 \dots w_8, \sigma)x - g(u, w_1 \dots w_8, \sigma) \quad (3.16)$$

is determined by the following three observations [23]:

1. The degrees of f and g in u are 4 and 6 respectively. This follows from the known behavior at large u .
2. The points w_k equal the integrals of the meromorphic two-form

$$\Omega = \frac{du \wedge dx}{y} \quad (3.17)$$

over eight 2-cycles of the total x - u space which generate an E_8 lattice.

3. The transformation:

$$y \rightarrow \alpha^3 y, \quad x \rightarrow \alpha^2 x, \quad u \rightarrow \alpha u + \beta \quad (3.18)$$

does not change the two-form Ω and hence does not change the prepotential.

4. From the large $|u|$ behavior it follows that

$$\begin{aligned} f(u) &= \frac{1}{256v_5^4} g_2(\sigma) u^4 + \mathcal{O}(u^3), \\ g(u) &= \frac{1}{2048v_5^6} g_3(\sigma) u^6 + \mathcal{O}(u^5). \end{aligned} \quad (3.19)$$

where

$$\begin{aligned} \frac{1}{4} g_2(\sigma) &= 15\pi^{-4} \sum_{m,n \in \mathbb{Z}_{\neq 0}} \frac{1}{(m\sigma + n)^4} \\ \frac{1}{4} g_3(\sigma) &= 35\pi^{-6} \sum_{m,n \in \mathbb{Z}_{\neq 0}} \frac{1}{(m\sigma + n)^6}. \end{aligned} \quad (3.20)$$

These four requirements completely determine f and g up to the transformation (3.18). The resulting functions are still in an implicit form, though. The explicit formulas are described in Appendix A.

We give below the formula for the curve corresponding to unbroken E_8 :

$$y^2 = x^3 - \frac{1}{256v_5^4} g_2(\sigma) u^4 x - (\alpha u^5 + \frac{1}{2048v_5^6} g_3(\sigma) u^6). \quad (3.21)$$

The coefficient of u^5 is arbitrary.

3.3. $d = 4$, $N = 2$, $SU(2)$ with $N_f = 4$ and its $SL(2, \mathbb{Z})$ duality

For generic Wilson lines, w_i , in (3.16) the E_8 singularity can split to ten singularities. Adding to these the two singularities discussed above (at $a = \frac{1}{\sqrt{2}R_6}$ and at $a_D = \frac{1}{\sqrt{2}R_5}$) there are 12 singularities. By tuning the Wilson lines to a special value we can group them into two groups of six singularities each. In the notation introduced in the next section these two singularities are of D_4 type. In terms of four dimensional field theory, the low energy theory near any of these singularities is $SU(2)$ with $N_f = 4$ flavors. A similar construction in the context of F-theory compactification to eight dimensions appeared in [29,14].

The $N = 2$, $SU(2)$ with $N_f = 4$ field theory is very special [24]. It is a finite quantum field theory with a dimensionless coupling constant τ . It was argued in [24] that this theory exhibits $SL(2, \mathbb{Z})$ duality—the theory with the parameter τ is the same as the theory with the parameter $\tau + 1$ or $-\frac{1}{\tau}$. This duality acts in a non-trivial way on the spectrum of the theory. Under $\tau \rightarrow \tau + 1$ monopoles transform into dyons and under $\tau \rightarrow -\frac{1}{\tau}$ electrons are interchanged with monopoles. This means that $SL(2, \mathbb{Z})$ acts on the global $Spin(8)$ symmetry as triality. The evidence for this duality presented in [24] was based on analyzing some of the spectrum and by examining the family of elliptic curves which determine the effective gauge coupling $\tau_{eff}(u)$ on the moduli space.

We now get a new understanding of this duality from our six-dimensional viewpoint. We are going to assume (as everywhere in this paper) that the six-dimensional E_8 theory exists. For large $|u|$ we see from (3.6), (3.7), and (3.8) that the effective coupling constant of the $U(1)$ theory is given by the complex structure of the torus we compactify on

$$\tau = \sigma. \tag{3.22}$$

Since the $SU(2)$ gauge theory with $N_f = 4$ is finite, this value of τ is not corrected in the quantum theory and therefore (3.22) is true everywhere in the moduli space.

Consider now labeling the torus by σ' which differs from σ by an $SL(2, \mathbb{Z})$ transformation. Clearly, this should not affect the effective theory. But from (3.22) it is clear that the value of τ of the $SU(2)$ theories is transformed by $SL(2, \mathbb{Z})$. Therefore, the $SL(2, \mathbb{Z})$ of the torus which is a “geometric symmetry” induces the $SL(2, \mathbb{Z})$ duality symmetry of the four dimensional field theory.

This $SL(2, \mathbb{Z})$ duality has already been related to the $SL(2, \mathbb{Z})$ action on T^2 in [30]. There, string compactification to four dimensions was studied and the finite $N = 2$ $SU(2)$

gauge theory appeared in the low energy approximation. The novelty of our construction is that we use fewer degrees of freedom—a six-dimensional field theory rather than string theory.

Our understanding of the duality of the four dimensional, finite $N = 2$ theory is similar to the understanding of the duality of the four dimensional $N = 4$ theory presented in [31]. In both cases a six dimensional field theory at a non-trivial fixed point of the renormalization group is compactified on a two torus with complex structure σ to four dimensions to yield a four dimensional field theory with $\tau = \sigma$. The duality of the four dimensional theory is then understood as a consequence of the $SL(2, \mathbb{Z})$ of the torus.

4. Singularities in four dimensions

In this section we examine what happens to the field theories of [4,5] upon compactification to four dimensions. Of course, they can be obtained as compactifications of the six-dimensional theory on \mathbf{T}^2 as $R_6 \rightarrow 0$.

All these theories have a moduli space which is topologically the complex u -plane. The coupling constant of the photon is determined by a torus fibered over this plane [21]. This torus also determines the metric on the moduli space. The singularities in such a setup were analyzed by Kodaira [25]. All of them have realizations in gauge theories:

1. A_n ($n = 0, 1, \dots$) singularities (corresponding to Kodaira's I_{n+1}). The monodromy around the singularity is $T^{n+1} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix}$ and the gauge coupling behaves like

$$\tau = \frac{n+1}{2\pi i} \log z + \mathcal{O}(1). \quad (4.1)$$

These theories arise in $U(1)$ gauge theories with $n+1$ electrons. The global symmetry of the theory is $SU(n+1)$ and the Higgs branch is the moduli space of $SU(n+1)$ instantons [24]. All these theories are IR free.

2. D_n ($n = 4, 5, \dots$) singularities (corresponding to Kodaira's I_{n-4}^*). The monodromy around the singularity is $PT^{n-4} = \begin{pmatrix} -1 & -n+4 \\ 0 & -1 \end{pmatrix}$: the gauge coupling behaves like

$$\tau = \frac{n-4}{2\pi i} \log z + \mathcal{O}(1), \quad (4.2)$$

and the periods behave like

$$\begin{aligned} a &= \sqrt{z} + \mathcal{O}(1) \\ a_D &= \frac{n-4}{\pi i} a \log a + \mathcal{O}(1). \end{aligned} \quad (4.3)$$

These theories arise in $SU(2)$ gauge theories with n quark hypermultiplets. The global symmetry of the theory is $SO(2n)$ and the Higgs branch is the moduli space of $SO(2n)$ instantons [24]. For $n > 4$ these theories are IR free and for $n = 4$ they are finite conformal field theories for all values of the bare coupling constant τ .

3. H_n ($n = 0, 1, 2$) singularities (corresponding to Kodaira's II, III, IV). τ is determined by the family of elliptic curves

$$\begin{aligned} y^2 &= x^3 - z && \text{for } H_0 \\ y^2 &= x^3 - zx && \text{for } H_1 \\ y^2 &= x^3 - z^2 && \text{for } H_2. \end{aligned} \tag{4.4}$$

The monodromies around these singularities are

$$\begin{aligned} (ST)^{-1} &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} && \text{for } H_0 \\ S^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} && \text{for } H_1 \\ (ST)^{-2} &= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} && \text{for } H_2. \end{aligned} \tag{4.5}$$

They arise in $SU(2)$ gauge theories with $n+1$ quark flavors by tuning the quark mass parameter to a special value [3] (the singularity H_0 was first found in $SU(3)$ gauge theories in [2]). The global symmetry of the H_n theory is $SU(n+1)$ (no symmetry for H_0) and a Higgs branch isomorphic to the moduli space of $SU(n+1)$ instantons (no Higgs branch for H_0). All of these are non-trivial conformal field theories. As (4.4) are deformed, they can split to $n+2$ A_0 singularities.

4. E_n ($n = 6, 7, 8$) singularities (corresponding to Kodaira's IV*, III*, II*). τ is determined by the family of elliptic curves

$$\begin{aligned} y^2 &= x^3 + z^4 && \text{for } E_6 \\ y^2 &= x^3 + xz^3 && \text{for } E_7 \\ y^2 &= x^3 + z^5 && \text{for } E_8. \end{aligned} \tag{4.6}$$

The monodromies around these singularities are

$$\begin{aligned} (ST)^2 &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} && \text{for } E_6 \\ S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} && \text{for } E_7 \\ ST &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} && \text{for } E_8. \end{aligned} \tag{4.7}$$

As we compactify the E_n ($n = 6, 7, 8$) five-dimensional theories on a circle, the E_n symmetry must be present in the four-dimensional theory and therefore we should find these theories. (These E_n cases were recently analyzed in [6,7,8].) The non-trivial conformal field theories associated with these have E_n global symmetry and the Higgs branch is the moduli space of E_n instantons.

In this classification we did not include what can be called D_n for $n = 0, 1, 2, 3$; i.e. the results of $SU(2)$ with n quark flavors. These theories are UV free and are strongly coupled at long distance. From the solution of these field theories [21,24] we learn that the singularities are

$$\begin{aligned} A_0 + A_3 & \quad \text{for } n = 3 \\ A_1 + A_1 & \quad \text{for } n = 2 \\ A_0 + A_0 + A_0 & \quad \text{for } n = 1 \\ A_0 + A_0 & \quad \text{for } n = 0. \end{aligned} \tag{4.8}$$

As a preliminary for matching the five-dimensional theories with the four-dimensional ones, let us examine the behavior for large $|u|$. The metric on the moduli space of the five-dimensional theory there is $(ds)^2 = (t_0 + \frac{c}{\sqrt{2}}\phi) d\phi d\phi$. The dimensional reduction leads to a factor of R_5 in this metric and to another compact scalar $\theta = R_5 A_5 \sim \theta + 2\pi$ with metric

$$(ds)^2 = R_5 \left(\frac{16\pi^2}{g_0^2} + \frac{c}{\sqrt{2}}\phi \right) d\phi d\phi + \frac{1}{8\pi^2 R_5} \left(\frac{16\pi^2}{g_0^2} + \frac{c}{\sqrt{2}}\phi \right) d\theta d\theta. \tag{4.9}$$

To cast it in $N = 2$ superspace we define

$$\begin{aligned} a &= \phi + \frac{i\theta}{2\sqrt{2}\pi R_5} = \frac{1}{2\sqrt{2}\pi R_5} \log u \\ a_D &= 2\pi i R_5 \left(\frac{8\pi}{g_0^2} a + \frac{c}{4\sqrt{2}\pi} a^2 \right) \end{aligned} \tag{4.10}$$

As in (3.14), we can work out the monodromy under $u \rightarrow e^{2\pi i} u$. This time it is non-trivial even in $SL(2, \mathbb{Z})$ —it is

$$\mathcal{M} = T^{c/2}. \tag{4.11}$$

In the A_n case (i.e., the $U(1)$ theories in five dimensions), c can be odd and this calculation is another indication that the $U(1)$ theory cannot be considered in isolation. However, if the $U(1)$ theory is embedded in another theory, the change in monodromy $T^{c/2}(T^{-c/2})^{-1} = T^c$ can be measured in that theory, and is sensible. In the D_n and E_n cases, c is even and this complication does not arise.

We can now consider the various five-dimensional theories:

1. $U(1)$ with n electrons. The moduli space \mathbf{R} becomes $\mathbf{R} \times \mathbf{S}^1$ in four dimensions where the \mathbf{S}^1 originates from the A_5 component of the gauge field. Classically, the metric is as in (4.9) with $c = 0$ and there are n massless electrons at $\phi = 0$. The one loop correction to the metric includes contributions of the massive Kaluza-Klein modes in the electrons and is given by a simple integral. Even without examining it explicitly, we can easily extract some of its properties. For large $|\phi|$ the metric and the monodromies are determined as in (4.9) and (4.11). Therefore, the monodromy around $\phi = 0$ is T^{-n} . This is consistent with the A_{n-1} singularity which is expected there. Indeed, as these theories are IR free both in five and in four dimensions, we do not expect the long distance behavior to change the tree level spectrum.
2. $SU(2)$ with n quarks. The moduli space \mathbf{R}/\mathbb{Z}_2 becomes $(\mathbf{R} \times \mathbf{S}^1)/\mathbb{Z}_2$ in four dimensions where the \mathbf{S}^1 originates from the A_5 component of the gauge field and \mathbb{Z}_2 from the Weyl group of $SU(2)$. Classically, the metric is as in (4.9) with $c = 0$. There are two special points. The obvious one is at $\phi = \theta = 0$: there is a four-dimensional $SU(2)$ theory with n quarks. Somewhat less obvious is the point $\phi = 0, \theta = \pi$, where there is an $SU(2)$ theory with no quarks. In terms of the variable u of (3.10) these two singularities are at $u = \pm 1$. Quantum mechanically this picture changes. First, at one loop the metric is corrected. The monodromy at large u is determined by (4.11) to be T^{8-n} . It is equal to the product of the monodromy around $\phi = \theta = 0$ which is PT^{4-n} and the monodromy around $\phi = 0, \theta = \pi$ which is PT^4 (where $P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL(2, \mathbb{Z})$). Non-perturbatively, the theory at $\phi = 0, \theta = \pi$ is always strongly coupled and the theory at $\phi = \theta = 0$ is strongly coupled for $n < 4$. Using (4.8), we determine the singularity structure in four dimensions for the reduction of the various five-dimensional theories

$$\begin{aligned}
D_n &\rightarrow D_n + A_0 + A_0 && \text{for } n \geq 4 \\
D_3 &\rightarrow A_0 + A_0 + A_0 + A_3 \\
D_2 &\rightarrow A_0 + A_0 + A_1 + A_1 \\
D_1 &\rightarrow A_0 + A_0 + A_0 + A_0 + A_0 \\
D_0 &\rightarrow A_0 + A_0 + A_0 + A_0 \\
\tilde{D}_0 &\rightarrow A_0 + A_0 + A_0 + A_0
\end{aligned} \tag{4.12}$$

where in the last one we used the fact that the long distance Lagrangian is the same as for D_0 —they differ only in a \mathbb{Z}_2 theta-like angle [22].

3. The non-trivial theories E_n . We have already seen in the previous section that the E_8 singularity splits

$$E_8 \rightarrow E_8 + A_0. \quad (4.13)$$

Similarly,

$$\begin{aligned} E_7 &\rightarrow E_7 + A_0 \\ E_6 &\rightarrow E_6 + A_0. \end{aligned} \quad (4.14)$$

One way to derive this is by perturbing the E_8 five-dimensional theory and examining the consequences in four dimensions. We can continue to flow down and determine what happens to the other non-trivial five-dimensional theories. E_5 has $SO(10)$ global symmetry and a Higgs branch which is the moduli space of $SO(10)$ instantons. There is only one theory in four dimensions with this Higgs branch and a one-dimensional Coulomb branch. This is the $SU(2)$ theory with $n = 5$ whose singularity is D_5 . Matching with the flow from E_6 in five dimensions we conclude

$$E_5 \rightarrow D_5 + A_0. \quad (4.15)$$

Since this theory is IR free, continuing to flow down is easy

$$\begin{aligned} E_4 &\rightarrow A_4 + A_0 + A_0 \\ E_3 &\rightarrow A_2 + A_1 + A_0 \\ E_2 &\rightarrow A_1 + A_0 + A_0 + A_0 \\ E_1 &\rightarrow A_1 + A_0 + A_0 \\ \widetilde{E}_1 &\rightarrow A_0 + A_0 + A_0 + A_0 \\ E_0 &\rightarrow A_0 + A_0 + A_0 \end{aligned} \quad (4.16)$$

4. Special values of the parameters. Here we can find the H_n singularities by starting with D_{n+1} in five dimensions with equal non-zero masses and tuning it to an appropriate value.

5. Applications to string theory

5.1. F-theory models in eight dimensions

In our first application to string theory of our analysis of the compactification to dimension four we encounter F-theory again, but in a different context. An F-theory model in eight dimensions can be regarded as a compactification of the type IIB string on a two-manifold B with some background D7-branes (generically, with 24 distinct D7-branes). The relation to our point of view is provided by the use of a D3-brane probe [14]. The field theory on a D3-brane probe has B as its moduli space. This can be seen quite explicitly as follows: on the one hand, the field theory is described in terms of the gauge coupling τ which depends on a family of elliptic curves; on the other hand, the same function τ is used to build the F-theory model, where it plays the role of the complexified dilaton on which $SL(2, \mathbb{Z})$ acts. The possible singularities in the family of elliptic curves—which correspond to possible ways that the D7-branes coalesce—are again described by Kodaira’s classification. In the notation we use here, these are A_n ($n \geq 0$), D_n ($n \geq 4$), E_n ($n = 6, 7, 8$) and H_n ($n = 0, 1, 2$). The A_n singularities correspond to the enhanced $SU(n+1)$ gauge symmetry from coalescing $n+1$ D7-branes [26,32,33], the D_n singularities correspond to the enhanced $SO(2n)$ gauge symmetry from coalescing n D7-branes and an orientifold plane [26,29,14] and the E_n singularities occur at strong coupling, as suggested in [34].

The splitting of singularities when compactifying from five to four dimensions can be seen directly in this application. The nine-dimensional type I’ compactification generically (on an open set in its moduli space) has two orientifold planes and 16 D8-branes.¹ The corresponding five-dimensional field theory content is 2 D_0 points and 16 A_0 points; in four dimensions, the D_0 ’s each split to 4 A_0 ’s, giving a total of 24 A_0 ’s, i.e., 24 D7-branes in F-theory.

Similar analyses can be made for non-generic singularities; to do so, we need to discuss gauge symmetry enhancement in these models. The possible ways in which the D8-branes can coalesce can be studied directly by considering the heterotic theories which are dual

¹ This is only one of three possible generic behaviors, each being valid on an appropriate open set in moduli [5]. The other two are (i) two orientifold planes, one of which is strongly coupled (giving an E_0 theory rather than a D_0 theory), and 17 D8-branes, and (ii) two strongly coupled orientifold planes (giving E_0 theories) and 18 D8-branes. All three possibilities lead to 24 A_0 points after further compactification.

to the type I' theories—the coalescing D8-branes correspond to gauge symmetry enhancement. In the heterotic interpretation, the root lattice of the gauge group must be embedded in the even unimodular lattice of signature $(1, 17)$, and for any such embedding, appropriate expectation values for the Wilson lines can be chosen to ensure that the gauge group is the one corresponding to such a sublattice.

When we further compactify on a circle, we must look for embeddings of the root lattice of the gauge group into the even unimodular lattice of signature $(2, 18)$. This can be seen either from the heterotic point of view, or from the F-theory point of view. In the F-theory version, we must find a K3 surface which has an elliptic fibration with a section, and some rational curves (contained in fibers of the fibration) whose intersection matrix reproduces the root system in question. Splitting off the classes of the base and fiber of the fibration from the cohomology lattice of the K3 leaves an even unimodular lattice of signature $(2, 18)$ into which the root lattice must be embedded. Thanks to the global Torelli theorem for K3 surfaces, for every such embedding there is a K3 surface which realizes it, so there is an F-theory model with the specified gauge symmetry group. It can always be realized by writing a Weierstrass equation in which the corresponding K3 surface has singular points of precisely the types specified by the root system. (The rational curves arise upon resolving those singularities.)

Note that the $(1, 17)$ lattice embeds into the $(2, 18)$ one, so any gauge group realized in nine dimensions is also realized in eight dimensions, consistent with the splitting of singularities we have discussed. There are effective techniques in the mathematics literature for determining whether a given root lattice has an embedding into the $(1, 17)$ or $(2, 18)$ lattice [35], and for determining the number of inequivalent embeddings [35, 36].

Although for any specified embedding of a root system into the $(2, 18)$ lattice, there is an F-theory realization with the corresponding gauge symmetry, this data does not completely determine the content of the four-dimensional field theory on the probe: a model with an H_n point is indistinguishable in this regard from a model in which the H_n point is replaced by an A_n point plus an A_0 point. For this reason, some care must be taken in applying lattice-embedding techniques to produce explicit models. For example, to determine if there exists a model with two E_8 points and an A_2 point, we note that the corresponding gauge group $E_8 \times E_8 \times SU(3)$ does occur on F-theory models [34]. (An explicit model was constructed in [34], but this fact can also be checked using lattice embedding techniques—the lattice embedding turns out to be unique.) The potential ambiguity between A_2 and H_2 points leads us to pursue a more explicit method. If we

begin with the Weierstrass equation which describes an F-theory model with gauge group containing $E_8 \times E_8$, given as in [17] by

$$y^2 = x^3 + \alpha x z^4 + z^5 + \beta z^6 + z^7, \quad (5.1)$$

and compute its discriminant

$$-z^{10} (27z^4 + 54\beta z^3 + (4\alpha^3 + 27\beta^2 + 54)z^2 + 54\beta z + 27). \quad (5.2)$$

then the E_8 singularities are at $z = 0$ and at $z = \infty$. If there is in addition an H_2 point or an A_2 point we can change coordinates to put it at $z = 1$; we then need a triple zero of the discriminant at that point. The only way to achieve that is to set $\alpha = 0$ and $\beta = -2$, leaving us with the equation

$$y^2 = x^3 + z^5(z - 1)^2, \quad (5.3)$$

which has an H_2 point at $z = 1$, not an A_2 point.

In [5] it was observed that there is a type I' model with an E_0 theory at each orientifold plane, and an A_{17} at another point, leading to $SU(18)/\mathbb{Z}_3$ enhanced gauge symmetry. (In fact, the existence of this theory was a crucial step in establishing the existence of the E_0 field theories.) In four dimensions, each E_0 point is expected to split to 3 A_0 points, leaving us with an F-theory model which (generically) has 6 A_0 points and one A_{17} point. Using lattice embedding techniques, the existence an F-theory model with $SU(18)/\mathbb{Z}_3$ in its gauge group can easily be established. (The quotient by \mathbb{Z}_3 corresponds in lattice-theoretic terms to the fact that the cokernel of the embedding map contains \mathbb{Z}_3 as its torsion subgroup; this is directly related to the need to use an exceptional modular invariant to construct the corresponding heterotic theory [5]. This phenomenon was also apparent in the $Spin(32)/\mathbb{Z}_2$ models constructed in [37].) To see that we actually get 6 A_0 points rather than some mixture of H_0 and A_0 points, we write an explicit model, with an equation of the form²

$$y^2 + (az^2 + bz + c)xy + z^6y = x^3, \quad (5.4)$$

² We should point out that the version of Tate's algorithm [38] which was formulated in [39] requires some care in its application to this case: because the order of vanishing of the discriminant is so large, the equation cannot be manipulated into the form specified in table 2 of [39] without introducing meromorphic changes of coordinates on the base B . In fact, when the discriminant of this equation is calculated, there are three miraculous cancellations of terms which allow the discriminant to vanish to order 18 at $z = 0$.

in which the $SU(18)$ occurs at $z = 0$. The discriminant of this cubic equation is

$$-\frac{1}{16}z^{18} (27z^6 - (az^2 + bz + c)^3), \quad (5.5)$$

so the other singularities are at the zeros of $27z^6 - (az^2 + bz + c)^3$, which generically has six distinct zeros (each giving an A_0 point). This confirms both the existence of the E_0 theory in five dimensions, and its splitting into 3 A_0 points in four dimensions.

Similar remarks apply to the $Spin(34)$ model of [5].

5.2. IIA theory compactified on a Calabi–Yau threefold

In our second application to string theory, we have five-dimensional models obtained by compactifying M-theory on a singular Calabi–Yau threefold, from which we can produce four-dimensional models by further compactification on a circle. In an appropriate domain in the moduli space these four-dimensional models can be interpreted as IIA string theory compactified on Calabi–Yau threefolds. The modifications to the moduli space arise from two sources: (1) the moduli includes a new 2-form in the NS-NS sector—the B -field—arising from zero-modes of the M-theory 3-form integrated along the circle, and (2) the moduli space is corrected by worldsheet instantons, which can be interpreted as the world-volume of the M-theory two-brane wrapping $S^1 \times \Sigma^{(i)}$ for surfaces $\Sigma^{(i)}$ in the Calabi–Yau manifold. The first modification complexifies the scalar in the vector multiplet, and the second is responsible for various quantum effects such as the splitting of singularities. We explore this latter point in detail below.

Let us first analyze these quantum effects on general grounds. Let $A_5^{(i)}$ be the area of the surface $\Sigma^{(i)}$, measured in M-theory units. The relationship between the M-theory scales and the type IIA scale and coupling then imply that

$$R_5 A_5^{(i)} \sim T^{(i)}, \quad (5.6)$$

where we have denoted the area of $\Sigma^{(i)}$ in type IIA units by $T^{(i)}$. Note that $T^{(i)}$ appears in a vector multiplet in the four dimensional field theory and hence the metric on the Kahler moduli space can depend on it. Another scalar made out of R_5 and the volume of the Calabi-Yau space in M theory units, S , is in a hypermultiplet and therefore cannot affect this metric.

We normalize the action so that worldsheet instantons of charge n contribute $e^{-2\pi n T^{(i)}}$ (multiplied by a phase); these can be interpreted as the wrapping of the M-theory two-brane worldvolume around $S^1 \times \Sigma^{(i)}$ (consistent with (5.6)). We let J and B denote the Kähler form and B -field, and introduce the notation

$$q^\sigma = e^{2\pi i \sigma \cdot (B + iJ)} \quad (5.7)$$

for the instanton contributions (where σ is the homology class of Σ) so that $q^{\sigma^{(1)}}, \dots, q^{\sigma^{(k)}}$ serve as coordinates on the Kähler moduli space.

If we pass to the field theory limit in five dimensions before compactifying, only a subset of the instantons will be available to correct the field theory correlation functions. In the Kähler moduli space, this corresponds to setting $q^{\sigma^{(j)}} = 0$ for any j for which $A_5^{(j)}$ decouples from the field theory. Typically this restricts the computation to the boundary of the Kähler moduli space, along which the Calabi–Yau manifold is singular. Techniques for computing in such limits were developed in [40].

In the A_n case, the worldsheet instanton sum associated to a flop transition was analyzed in [41,40]. Performing a flop on $n+1$ rational curves (all from the same homology class) alters the intersection numbers of divisors on the Calabi–Yau threefold, which is why the five-dimensional gauge coupling is only a piecewise linear function. However, the corrections from wrapped two-branes modify the singularity so that the coupling in four dimensions has a pole rather than a discontinuous derivative [18].

A typical correlation function is given by the intersection number $H_1 \cdot H_2 \cdot H_3$ in the M-theory context. (The gauge coupling can be determined from the behavior of $H \cdot H \cdot H$ for appropriate divisors H [42,43,44,18,5].) If γ is the common homology class of the $n+1$ rational curves being contracted, this correlation function is corrected by instantons to the following value in the four-dimensional theory

$$\langle H_1 H_2 H_3 \rangle = H_1 \cdot H_2 \cdot H_3 + (n+1) \frac{q^\gamma}{1 - q^\gamma} (\gamma \cdot H_1)(\gamma \cdot H_2)(\gamma \cdot H_3), \quad (5.8)$$

valid for small $|q^\gamma|$. (We have suppressed all instanton corrections other than the ones related to γ , which is the only relevant class in the field theory.) The expression (5.8) for the coupling can be analytically continued past the singularity at $q^\gamma = 1$, by employing the identity

$$\frac{q^\gamma}{1 - q^\gamma} = -1 - \frac{q^{-\gamma}}{1 - q^{-\gamma}} \quad (5.9)$$

to yield

$$\begin{aligned} \langle H_1 H_2 H_3 \rangle &= (H_1 \cdot H_2 \cdot H_3 - (n+1)(\gamma \cdot H_1)(\gamma \cdot H_2)(\gamma \cdot H_3)) \\ &\quad + (n+1) \frac{q^{-\gamma}}{1 - q^{-\gamma}} (-\gamma \cdot H_1)(-\gamma \cdot H_2)(-\gamma \cdot H_3). \end{aligned} \quad (5.10)$$

The first term in (5.10) is the M-theory coupling (i.e., the classical intersection product) on the flopped Calabi–Yau manifold

$$\widehat{H}_1 \cdot \widehat{H}_2 \cdot \widehat{H}_3 = H_1 \cdot H_2 \cdot H_3 - (n+1)(\gamma \cdot H_1)(\gamma \cdot H_2)(\gamma \cdot H_3), \quad (5.11)$$

and the entire expression (5.10) is seen to be precisely the instanton-corrected value for $\langle \widehat{H}_1 \widehat{H}_2 \widehat{H}_3 \rangle$ (bearing in mind that the homology class of the instanton changes sign during the flop).

In the D_n case, the instantons shrinking to zero size are a bit more complicated. Let 2γ be the homology class of the fiber of the ruled surface S which is shrinking, so that γ is the class of any of the components of the singular fibers in that ruling. We have n singular fibers in the ruling, each with 2 components, so there are a total of $2n$ instantons in class γ . On the other hand, in the homology class 2γ there is an entire \mathbf{CP}^1 of shrinking rational curves which according to the calculations of [45] contribute an instanton number of -2 . There are no instantons in any other multiples of γ . Thus, the entire instanton correction to $H_1 \cdot H_2 \cdot H_3$ due to the holomorphic curves which shrink to zero size at the D_n point is

$$\begin{aligned} &2n \frac{q^\gamma}{1 - q^\gamma} (\gamma \cdot H_1)(\gamma \cdot H_2)(\gamma \cdot H_3) + -2 \frac{q^{2\gamma}}{1 - q^{2\gamma}} (2\gamma \cdot H_1)(2\gamma \cdot H_2)(2\gamma \cdot H_3) \\ &= \left(2n \frac{q^\gamma}{1 - q^\gamma} - 2 \frac{8q^{2\gamma}}{1 - q^{2\gamma}} \right) (\gamma \cdot H_1)(\gamma \cdot H_2)(\gamma \cdot H_3) \\ &= \left((2n - 8) \frac{q^\gamma}{1 - q^\gamma} - 8 \frac{q^\gamma}{-1 - q^\gamma} \right) (\gamma \cdot H_1)(\gamma \cdot H_2)(\gamma \cdot H_3). \end{aligned} \quad (5.12)$$

This computation clearly shows the splitting of the five-dimensional singularity to two singularities in four dimensions, at $q^\gamma = \pm 1$, whose locations only differ by the value of the γ -component of the B -field, which is 0 in one case and $1/2$ in the other. This feature of D_n -type Calabi–Yau theories does not seem to have been observed before.

As in [5], the gauge coupling in five dimensions is determined by $S \cdot S \cdot (\phi S + t_0 H_0)$ where H_0 is a divisor meeting S in a section of the ruling; this is the second derivative

of the prepotential \mathcal{F} with respect to the parameter ϕ associated to S . The quantum corrected version of the *third* derivative is given by

$$\frac{\partial^3 \mathcal{F}}{\partial \phi^3} = \langle S S S \rangle = (8 - n) + (-1)^3 \left((2n - 8) \frac{q^\gamma}{1 - q^\gamma} - 8 \frac{q^\gamma}{-1 - q^\gamma} \right), \quad (5.13)$$

suppressing other instanton corrections. We identify the period a in the field theory with the area of γ , (i.e., $a = -\phi$ since $\gamma \cdot S = -1$), and the period a_D as

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = -\frac{\partial \mathcal{F}}{\partial \phi}; \quad (5.14)$$

we also have $q^\gamma = e^{-2\pi i \phi}$. Then the leading order behavior of $\partial^3 \mathcal{F} / \partial \phi^3$ near $q^\gamma = 1$ is

$$\frac{\partial^3 \mathcal{F}}{\partial \phi^3} = -\frac{2n - 8}{2\pi i \phi} + \dots, \quad (5.15)$$

so the leading order behavior of a_D is

$$a_D = \frac{2n - 8}{2\pi i} a \log a + \dots. \quad (5.16)$$

The generator of the monodromy is described as in [45] by the *elementary transformation* associated to the ruled surface S [46,47]. That transformation is a map on cohomology

$$\rho(H) = H + (2\gamma \cdot H)S, \quad (5.17)$$

which is a kind of reflection in the class S , mapping S to $-S$. It thus acts on ϕ and on a as multiplication by -1 ; combining with (5.16) we see that the monodromy action agrees with (4.3), as expected for a D_n point.

This same elementary transformation generates monodromy about $q^\gamma = -1$ as well, since $\phi \mapsto -\phi \pmod{\mathbb{Z}}$ has fixed points at both 0 and $1/2$. The monodromy at $1/2$ is exactly as expected for a D_0 point, by the same computation. We confirm in this way the splitting of the D_n point from dimension five into a D_n and a D_0 point in dimension four.

In this computation, we have suppressed instantons which wrap other holomorphic curves in the surface which shrinks to zero size. That is a valid approximation if the base of the ruled surface is taken to be extremely large—in this approximation, the quantum corrections to the four-dimensional field theory are suppressed so we see the “classical” splitting into two singularities without the further “quantum” splitting of one or both of those singularities. To see the full description we should consider these other instantons.

Let $\Sigma^{(1)}$ be the base \mathbf{CP}^1 of the ruled surface. Then the five-dimensional gauge coupling for the D_n theory is given by

$$\frac{1}{g_5^2} \sim A_5^{(1)}, \quad (5.18)$$

and the four-dimensional gauge coupling is

$$\frac{1}{g_4^2} \sim R_5 A_5^{(1)} \sim T^{(1)}. \quad (5.19)$$

We introduce the homology class η of $\Sigma^{(1)}$ so that the corresponding instanton contribution is q^η .

From the solution to the D_0 field theory, we predict a splitting of the singularity at $\gamma \cdot B = 1/2$ of the form

$$u_+ - u_- \sim q^{\eta/2}. \quad (5.20)$$

That is, including the instantons wrapping the base \mathbf{CP}^1 should modify the appropriate term $-8q^\gamma/(-1 - q^\gamma)$ from (5.12) to something of the form

$$\begin{aligned} & \frac{-4q^\gamma + \mathcal{O}(q^{\eta/2})}{-1 - q^\gamma + q^{\eta/2}f(q^\gamma) + \mathcal{O}(q^\eta)} + \frac{-4q^\gamma + \mathcal{O}(q^{\eta/2})}{-1 - q^\gamma - q^{\eta/2}f(q^\gamma) + \mathcal{O}(q^\eta)} \\ &= \frac{8q^\gamma(1 + q^\gamma) + \mathcal{O}(q^\eta)}{(1 + q^\gamma)^2 - q^\eta f(q^\gamma)^2 + \mathcal{O}(q^{2\eta})} \end{aligned} \quad (5.21)$$

for some function $f(q^\gamma)$ which does not vanish at $q^\gamma = -1$. (The terms $\mathcal{O}(q^{\eta/2})$ and $\mathcal{O}(q^\eta)$ also depend on q^γ .)

The denominator in (5.21) determines the location of a component of the discriminant locus in the quantum-corrected (vector) moduli space of the Calabi–Yau manifold. There is another, related subset of the boundary of the moduli space: the locus where $q^\eta = 0$ and all $q^{\sigma^{(j)}} = 0$ for $\sigma^{(j)} \neq \eta, \gamma$; we have identified this locus with the weak coupling limit in the field theory. It is a holomorphic curve which is tangent to the discriminant-locus component $(1 + q^\gamma)^2 - q^\eta f(q^\gamma)^2 + \mathcal{O}(q^{2\eta}) = 0$ defined by (5.21).

This qualitative feature of an asymptotically free $SU(2)$ theory—a component of the discriminant locus which is tangent to a curve in the boundary, due to the quantum splitting of singularities—was presented by Kachru and Vafa [48] as evidence that they had correctly identified a four-dimensional heterotic/type II dual pair [48,30]. The behavior near such tangent loci was subsequently investigated in greater detail [49,50]. Here we observe that such a structure will be a generic feature in Calabi–Yau moduli space any time a rational

ruled surface shrinks to zero size. (Some related observations have been made in [51].) This can be seen even in models with no manifest heterotic dual [52,51]. The compatibility with heterotic/type II duality (if it is present) becomes clear when one recalls that for Calabi–Yau models with a heterotic dual, there is always a K3 fibration over a base Σ , with the area of Σ mapping to the heterotic coupling [53]; moreover, a perturbative $SU(2)$ on the heterotic side corresponds [54] to a ruled surface on the Calabi–Yau threefold over the same base Σ .

When $n \leq 3$, the singularity at $\gamma \cdot B = 0$ should exhibit a similar behavior. That is, the other term $(2n - 8)q^\gamma/(1 - q^\gamma)$ from (5.12) should be modified by instantons to something of the approximate form

$$\begin{aligned} & \frac{(n - 4)q^\gamma + \mathcal{O}(q^{\eta/2})}{1 - q^\gamma + q^{\eta/2}g(q^\gamma) + \mathcal{O}(q^\eta)} + \frac{(n - 4)q^\gamma + \mathcal{O}(q^{\eta/2})}{1 - q^\gamma - q^{\eta/2}g(q^\gamma) + \mathcal{O}(q^\eta)} \\ &= \frac{(2n - 8)q^\gamma(1 - q^\gamma) + \mathcal{O}(q^\eta)}{(1 - q^\gamma)^2 - q^\eta g(q^\gamma)^2 + \mathcal{O}(q^{2\eta})} \end{aligned} \quad (5.22)$$

for some function $g(q^\gamma)$ which does not vanish at $q^\gamma = -1$.

To summarize our conclusions about the D_n case: a rational ruled surface shrinking to a curve is associated to two components of the discriminant locus,³ one of which is tangent to the weak coupling limit locus $q^\eta = 0$, $q^{\sigma^{(j)}} = 0$. The other component will also be tangent to that locus if $n \leq 3$. This entire structure is associated to a *single* $SU(2)$ factor of the gauge group. The non-trivial dynamics in four dimensions are responsible for the somewhat indirect way in which this $SU(2)$ manifests itself—it is more clearly visible in five dimensions.

We turn now to the E_n case, in which a surface S shrinks to a point. We assume that we are in the generic situation in which the image of $H^{1,1}(X) \rightarrow H^{1,1}(S)$ is one-dimensional. The coefficient c which governs the five-dimensional field theory is calculated by the intersection product $S \cdot S \cdot S$. In four dimensions, this is corrected by instantons to

$$\langle S S S \rangle = (S \cdot S \cdot S) + \sum_{j=1}^{\infty} N_j \frac{j^3 q^{j\gamma}}{1 - q^{j\gamma}}, \quad (5.23)$$

³ There are two distinct components near the weak-coupling limit, but they could be globally identified in some examples. In addition, it might happen that the map from the field theory moduli space to the Calabi–Yau moduli space is many-to-one, and these components could in principle have the same image.

where γ is a homology class on S such that $S \cdot \gamma = -1$, and where N_j is the instanton number associated with rational curves in class $j\gamma$. In these E_n cases, unlike the previous two, there will be an infinite number of homology classes contributing to the instanton sum, but the answer is universal for the E_n theory in question (depending only on n —of course there is also a sum for \tilde{E}_1). The number N_1 should be the number of “lines” on S , that is, the number of \mathbf{CP}^1 ’s whose “degree” $-S \cdot \gamma = c_1(S) \cdot \gamma$ is 1. The numbers N_j for $j > 1$ have a less straightforward interpretation, since the family of \mathbf{CP}^1 ’s in such a class usually has positive dimension.

In the case of E_0 , the first several terms of this instanton expansion (5.23) were calculated in [55,56]:

$$\begin{aligned} 9 + 3 \frac{3^3 q^{3\gamma}}{1 - q^{3\gamma}} - 6 \frac{6^3 q^{6\gamma}}{1 - q^{6\gamma}} + 27 \frac{9^3 q^{9\gamma}}{1 - q^{9\gamma}} - 192 \frac{12^3 q^{12\gamma}}{1 - q^{12\gamma}} \\ + 1695 \frac{15^3 q^{15\gamma}}{1 - q^{15\gamma}} - 17064 \frac{18^3 q^{18\gamma}}{1 - q^{18\gamma}} + 188454 \frac{21^3 q^{21\gamma}}{1 - q^{21\gamma}} - \dots, \end{aligned} \quad (5.24)$$

and the meaning of the higher terms was explored in [56], where a clear geometric interpretation was found for the first two nonzero terms $N_3 = 3$, $N_6 = -6$. It was also noted there that many of the higher terms, beginning with N_6 , are negative. In fact, the series appears to alternate signs after the first term.

The splitting of E_0 into three singularities in four dimensions has a clear interpretation from this point of view. For the del Pezzo surface $S = \mathbf{CP}^2$ associated to the E_0 theory, N_j must vanish unless j is divisible by 3. This is because $c_1(S)$ has intersection number 3 with the generator of second homology. Thus, the entire series (5.24) is a function of $q^{3\gamma}$, and it will have three poles near the origin whose values differ by a cube root of unity. At those singularities, the value of the area $\gamma \cdot J$ will be shifted away from zero [40] and the γ -component of the B -field $\gamma \cdot B$ will take one of three possible values 0, $1/3$ or $2/3$.⁴

For the higher E_n ’s some explicit calculations of (5.23) were made for the cases of E_5 , E_6 , E_7 , E_8 in [27]. There is a mysterious discrepancy [27] between the first coefficient and the number of lines in the case of E_8 , but in the other cases the number of lines corresponds to N_1 as expected. These series also appear to alternate signs after the first term. It will be very interesting to study these series further in order to verify other aspects of our qualitative description.

⁴ For some examples, including the case discussed in [40], there will be a three-to-one map from the field theory moduli space to the Calabi–Yau moduli space, which identifies these three singularities, so that only the value $\gamma \cdot B = 0$ occurs.

6. Compactification to three dimensions

The compactification of all these theories to three dimensions is easily analyzed either by using string duality as in [15] or using field theory methods as in [57]. This section is mostly a review of [15,57] in the notation of this paper.

In three dimensions the Coulomb branch is a hyper-Kähler manifold. In our case it is of real dimension four. Three of the coordinates of this manifold arise from the compactification of the six-dimensional two-form. The fourth one is dual to the three-dimensional vector.

The Coulomb branch for these theories is as follows.

$U(1)$ gauge theory with n electrons, A_{n-1} : The Coulomb branch has an A_{n-1} singularity. In particular, for $n = 1$ it is smooth. The theories at the singularities are at non-trivial fixed points.

$SU(2)$ gauge theory with n quarks, D_n : The Coulomb branch has a D_n singularity. In particular, for $n = 0, 1$ it is smooth, for $n = 2$ it has two A_1 singularities and for $n = 3$ it has an A_3 singularity. The theories at the singularities are at non-trivial fixed points.

In [57] these four-dimensional gauge theories were studied on $\mathbf{R}^3 \times \mathbf{S}^1$ as a function of the radius R_4 of \mathbf{S}^1 . The moduli space of the theory on \mathbf{R}^4 has complex dimension one and has an auxiliary torus fibered over it, which determines the metric on the moduli space. The moduli space of the theory on $\mathbf{R}^3 \times \mathbf{S}^1$ is the full four (real) dimensional fiber bundle where the area of the auxiliary torus is $\frac{1}{R_4}$.

These facts can be derived by considering the effective theory of a two-brane in compactification of M-theory on K3. Since the brane is at a point in K3, the moduli space of vacua of the theory on the brane is K3. For special values of the space time moduli (the parameters of the theory) this moduli space becomes singular. These singularities are classified by an ADE classification. Using this fact we can determine the fate of the various singularities in four dimensions upon compactification

$$\begin{aligned}
A_n &\rightarrow A_n && \text{for } n \neq 0 \\
D_n &\rightarrow D_n && \text{for } n \geq 4 \\
E_n &\rightarrow E_n && \text{for } n = 6, 7, 8 \\
H_n &\rightarrow A_n && \text{for } n = 1, 2.
\end{aligned} \tag{6.1}$$

(The A_0 and H_0 points are nonsingular in dimension three.)

Notice that this is precisely the same mathematical phenomenon which was responsible for the ambiguities in the lattice-theoretic specification of F-theory models: only the singularity in the total space of the K3 surface, not the type of the singular fiber in the fibration, matters in determining the gauge group (in the earlier example) or the three-dimensional moduli space (in the present example).

In higher dimensions there are no known free field theories which flow to the E_n theories. This fact has led some authors to suggest that they are not local quantum field theories. At least in three dimensions, such free field theories were found in [58]. Hence these theories have a Lagrangian description and they are clearly local quantum field theories.

One can generalize this discussion by considering the compactification of the six-dimensional theory on \mathbf{T}^3 as a function of its parameters and background ($B_{\mu\nu}$ $\mu, \nu = 4, 5, 6$). All these parameters are in vector multiplets and therefore, as explained in the introduction, various non-renormalization theorems apply, e.g., the metric on the Higgs branch is independent of these parameters.⁵ The appearance of these theories as effective field theories on branes makes it obvious that the resulting moduli space is again a piece of a K3 whose moduli depend on the quark masses and the parameters of the compact \mathbf{T}^3 . Therefore, by changing the parameters of the \mathbf{T}^3 we can explore the dynamics of these theories in various dimensions. The answer is always a piece of a K3. It is amazing that the same object (K3) provides the answer to so many different quantum field theories in different dimensions!

Acknowledgments

The work of O.J.G. was supported in part by a Robert H. Dicke fellowship and by DOE grant DE-FG02-91ER40671; that of D.R.M. was supported in part by the Harmon Duncombe Foundation and by NSF grants DMS-9401447 and DMS-9627351; and that of N.S. was supported in part by DOE grant DE-FG02-96ER40559. We thank P. Berglund, B. Greene, A. Hanany, A. Losev, G. Moore, V. Nikulin, Y. Oz, R. Plesser, S. Shatashvili, S. Shenker, and E. Witten for helpful discussions.

⁵ In three dimensions vector multiplets are dual to hypermultiplets. Nevertheless, the non-renormalization theorems can be used because the scalars in these multiplets transform under different global $SU(2)$ symmetries which are exchanged by the duality transformation [58].

Appendix A. Formulas for generic Wilson lines

In section (3.2) we described the procedure to obtain the 4D Seiberg-Witten curve corresponding to compactification with arbitrary E_8 twists along \mathbf{T}^2 . The curve was given in implicit form. It is the purpose of this appendix to express the coefficients of f and g as functions of the twists.

To describe the E_8 twists we decompose

$$SO(2)^8 \subset SO(16) \longleftarrow Spin(16) \subset E_8. \quad (\text{A.1})$$

and let ρ_1, \dots, ρ_8 be the values of the $SO(2)^8$ Wilson lines around R_6 and let $\omega_1, \dots, \omega_8$ be the values around R_5 .

We define the complex variables:

$$w_k = \omega_k + \rho_k \sigma, \quad k = 1 \dots 8. \quad (\text{A.2})$$

These are 8 points on the torus. The prepotential depends on them in a holomorphic manner, subject to periodicity and to the E_8 Weyl group identifications:

$$\begin{aligned} (w_1, \dots, w_8) &\sim (w_1 + \frac{1}{2}n_1 + \frac{1}{2}m_1\sigma, \dots, w_8 + \frac{1}{2}n_8 + \frac{1}{2}m_8\sigma), \\ n_i, m_i &\in \mathbb{Z}, \quad \sum_1^8 n_i \equiv \sum_1^8 m_i \equiv 0 \pmod{2} \\ (w_1, \dots, w_8) &\sim (w_{\psi(1)}, \dots, w_{\psi(8)}), \quad \psi \in S_8 \\ (w_1, \dots, w_8) &\sim ((-1)^{\epsilon_1}w_1, \dots, (-1)^{\epsilon_8}w_8), \quad \sum_1^8 \epsilon_i \equiv 0 \pmod{2} \\ (w_1, \dots, w_8) &\sim (w_1 - \frac{\sum_1^8 w_i}{4}, \dots, w_8 - \frac{\sum_1^8 w_i}{4}). \end{aligned} \quad (\text{A.3})$$

With these Wilson lines the central charge formula is

$$M = \sqrt{2}|n_e a + n_m a_D - 2\pi i v_5(P_5 + \sum_{i=1}^8 S_i \omega_i) + 2\pi i v_6(P_6 + \sum_{i=1}^8 S_i \rho_i)|. \quad (\text{A.4})$$

Here P_5 and P_6 are the momenta around R_5 and R_6 measured in integer units and S_i are the the integer $SO(2)$ charges in (A.1).

We recall that ω_i and ρ_i , being Wilson lines, were defined only up to integer shifts. Indeed (A.4) is invariant under a change

$$\omega_i \rightarrow \omega_i + n_i, \quad P_5 \rightarrow P_5 - \sum_{i=1}^8 n_i S_i \quad (\text{A.5})$$

for integer n_i 's.

Also, under a modular transformation

$$R_5 \leftrightarrow R_6, \quad \varphi \rightarrow \pi - \varphi, \quad \omega_i \rightarrow -\rho_i, \quad \rho_i \rightarrow \omega_i, \quad P_5 \rightarrow -P_6, \quad P_6 \rightarrow P_5, \quad (\text{A.6})$$

the formula for the mass is still invariant provided we change a and a_D according to

$$a \rightarrow -ia_D, \quad a_D \rightarrow ia, \quad n_e \rightarrow n_m, \quad n_m \rightarrow -n_e. \quad (\text{A.7})$$

which is consistent with (3.11).

The total space described by the elliptic fibration (3.16) is the almost Del Pezzo surface with $\chi = 12$ that also appeared in the F-theory construction of E_8 tensionless strings [17]. An alternative, more convenient, description of it is given by the blow-up of \mathbf{CP}^2 at 9 points e_0, \dots, e_8 which have to be the intersection of two cubics. Let the homogeneous coordinates on \mathbf{CP}^2 be X, Y, Z and let the two cubics be

$$P(X, Y, Z) = 0, \quad Q(X, Y, Z) = 0. \quad (\text{A.8})$$

Then there is a whole family of cubics which intersect at $e_0 \dots e_9$:

$$Q(X, Y, Z) + uP(X, Y, Z) = 0 \quad (\text{A.9})$$

where u is a coordinate on \mathbf{CP}^1 . The equation (A.9) exhibits the elliptic fibers of \mathbf{CP}^2 blown-up at the 9 points.

The cohomology structure is as follows [17]: the class of the fiber is

$$f = 3H - \sum_{i=1}^9 e_i \quad (\text{A.10})$$

where H is the hyperplane section of \mathbf{CP}^2 . The section of the elliptic fibration can be chosen to be the exceptional divisor e_0 and the E_8 is generated by

$$H - e_1 - e_2 - e_3, \quad e_k - e_0. \quad (\text{A.11})$$

Now returning to (3.16) we have to integrate the meromorphic two-form Ω over the E_8 basis. It will be more convenient to integrate the two-form over $e_i - e_0$. The difference is just a linear combination (using the fact that the integral of Ω on a fiber vanishes so that we can subtract $\frac{1}{3}f$ from $H - e_1 - e_2 - e_3$). Now we need to identify Ω in the \mathbf{CP}^2

variables. This is done by noting that Ω has a single simple pole on the whole fiber at $u = \infty$ which sets

$$\Omega = \frac{d(X/Z) \wedge d(Y/Z)}{P(X/Z, Y/Z, 1)}.$$

Now we have to calculate

$$\int_{[e_i] - [e_0]} \Omega.$$

The e_i 's are analytic classes, so the integral of a general $(2, 0)$ -form on them would vanish. However, the e_i all lie on the curve $P = 0$ where Ω has a pole. If we perform a linear change of variables to set

$$P(X, Y, Z) = X^3 - \frac{1}{4}g_2(\sigma)XZ^2 - \frac{1}{4}g_3(\sigma)Z^3 - Y^2Z \quad (\text{A.12})$$

where we used that fact that at $u = \infty$ the modulus of the cubic has to be σ , the integral turns out to be

$$w_i = \int_{e_0}^{e_i} \frac{dx'}{y'}$$

where $x' = X/Z$ and $y' = Y/Z$. In other words, under a map Φ from the elliptic curve $P = 0$ to a lattice $\mathbf{C}/\mathbb{Z} + \mathbb{Z}\sigma$ the points e_i get mapped to $\Phi(e_0) + w_i$. If we only knew what $\Phi(e_0)$ is we could have completed the calculation. To determine that we note

$$\sum_{i=0}^8 \Phi(e_i) = 0 \pmod{\mathbb{Z} + \mathbb{Z}\sigma}.$$

The standard proof is as follows [59]: A general cubic in \mathbf{CP}^2 intersects $P = 0$ at 9 points so we can define a map from the space of cubics in \mathbf{CP}^2 into $\mathbf{C}/\mathbb{Z} + \mathbb{Z}\sigma$ by the sum of Φ of the intersection points. The space of all cubics, however, is isomorphic to \mathbf{CP}^9 and the only holomorphic map from \mathbf{CP}^9 to a torus is a constant because it is a constant when restricted to every line $\mathbf{CP}^1 \subset \mathbf{CP}^9$.

So we find that

$$\begin{aligned} e_i &= (X = \frac{1}{\pi^2} \wp(\Phi(e_i)), Y = \frac{2}{\pi^3} \wp'(\Phi(e_i)), Z = 1) \\ w_i &= \Phi(e_i) - \Phi(e_0) \\ 0 &= \sum_{i=0}^8 \Phi(e_i) \end{aligned} \quad (\text{A.13})$$

Let's denote

$$\xi_i \equiv \Phi(e_i), \quad i = 0 \dots 8. \quad (\text{A.14})$$

The ξ_i 's can easily be calculated, given the w_i 's.

So, given σ we know $P(X, Y, Z)$ from (A.12) and given the w_i 's we can calculate the ξ_i 's from (A.13) and then we can find $Q(X, Y, Z)$ to be the cubic that passes through the e_i 's.

We write

$$Q(X_1, X_2, X_3) = \sum_{i,j,k} Q_{ijk} X_i X_j X_k \quad (\text{A.15})$$

where the Q_{ijk} satisfy (putting $Q_{111} = 0$ so as not to get (A.12) back):

$$\begin{aligned} 0 = & Q_{333} + \frac{8}{\pi^9} Q_{222} \wp'(\xi_i)^3 + \frac{6}{\pi^7} Q_{112} \wp(\xi_i)^2 \wp'(\xi_i) + \frac{3}{\pi^4} Q_{113} \wp(\xi_i)^2 \\ & + \frac{12}{\pi^6} Q_{223} \wp'(\xi_i)^2 + \frac{12}{\pi^8} Q_{122} \wp'(\xi_i)^2 \wp(\xi_i) + \frac{3}{\pi^2} Q_{133} \wp(\xi_i) \\ & + \frac{6}{\pi^3} Q_{233} \wp'(\xi_i) + \frac{12}{\pi^5} Q_{123} \wp(\xi_i) \wp'(\xi_i) \end{aligned} \quad (\text{A.16})$$

Those are 8 equations $i = 1 \dots 8$ for 9 variables so there is one more solution.

The family of curves now has the form

$$\begin{aligned} 0 = & u(X^3 - f_4 X Z^2 - g_2 Z^3 - Y^2 Z) + (Q_{333} Z^3 + Q_{222} Y^3 + 3Q_{112} X^2 Y + 3Q_{113} X Y^2 \\ & + 3Q_{223} Y^2 Z + 3Q_{122} Y^2 X + 3Q_{133} X Z^2 + 3Q_{233} Y Z^2 + 6Q_{123} X Y Z). \end{aligned} \quad (\text{A.17})$$

It can be changed into the form (3.16) by an $SL(3, \mathbf{C})$ transformation. The final step is to go back to the dimensionful variables

$$x = \frac{1}{8v_5^2} \left(\frac{X}{Z} \right), \quad y = \frac{1}{16\sqrt{2}v_5^3} \left(\frac{Y}{Z} \right). \quad (\text{A.18})$$

Appendix B. Decompactification from 4D to 5D

The decompactification limit is reached by taking $R_5 \rightarrow \infty$ leaving R_6 fixed. This means that $q \rightarrow 0$, but at the same time we must scale u according to

$$u = q^{-\psi}, \quad \psi = \sqrt{2} R_6 \phi, \quad (\text{B.1})$$

where ψ is kept fixed. The motivation for this is that in the large tension limit,

$$|u| \sim e^{4\pi^2 R_5 R_6 \times \text{Tension}}. \quad (\text{B.2})$$

In the decompactification limit we expect the 4D $U(1)$ coupling constant to look like

$$\tau = \frac{8\pi i}{g_5^2} = iR_5 F(\psi) \quad (\text{B.3})$$

where $F(\psi)$ is a piecewise linear function of ψ [4].

Such a behavior will be obtained as follows. For very large τ we can extract τ from (3.16):

$$\frac{f^3}{g^2} = \frac{27}{4} + 11664e^{2\pi i\tau} + \dots$$

On the other hand, f and g are given by

$$f = \sum_{j=0}^4 f_j u^j, \quad g = \sum_{l=0}^6 g_l u^l. \quad (\text{B.4})$$

For small q the coefficients f_4 and g_6 are:

$$\begin{aligned} f_4 &= \frac{1}{256v_5^4} g_2(\sigma) = \frac{1}{256v_5^4} \left(\frac{1}{3} + 80q + \mathcal{O}(q^2) \right) \\ g_6 &= \frac{1}{2048v_5^6} g_3(\sigma) = \frac{1}{2048v_5^6} \left(\frac{2}{27} - \frac{112}{3}q + \mathcal{O}(q^2) \right) \end{aligned} \quad (\text{B.5})$$

The other coefficients will also behave like $q^{-\alpha_k}$ for some powers α_k which depend on the Wilson lines. Thus we will find an expression of the form

$$\frac{f^3}{g^2} - \frac{27}{4} = 11664q + c_1 q^{\psi-\alpha_1} + c_2 q^{2\psi-\alpha_2} + \dots \quad (\text{B.6})$$

For very large ψ the first term on the right is dominant as $q \rightarrow 0$ giving a constant τ in 5D. As we decrease ψ we will reach a value for which one of the other terms becomes dominant. This way we see that τ is a piecewise linear function of ψ with jumps in the derivative where we switch from one term to the other in (B.6).

B.1. Decompactification for unbroken E_8

Now let's take the limit $q \rightarrow 0$ in (3.21). Substituting (B.1) in (3.21) and using (B.5) we find:

$$\tau = \begin{cases} \sigma & \text{for } \psi > 1 \\ \psi\sigma + \mathcal{O}(\log|\sigma|) & \text{for } 1 > \psi > 0 \\ \frac{1}{2} + i\frac{\sqrt{3}}{2} & \text{for } 0 > \psi \end{cases} \quad (\text{B.7})$$

The 5D coupling constant is thus:

$$\frac{8\pi}{g_5^2} = \lim_{R_5 \rightarrow \infty} \left(\frac{\tau}{2\pi i R_5} \right) = \begin{cases} \frac{1}{2\pi R_6} & \text{for } \psi > 1 \\ \frac{1}{2\pi R_6} \psi & \text{for } 1 > \psi > 0 \\ 0 & \text{for } 0 > \psi \end{cases} \quad (\text{B.8})$$

which agrees with (2.8).

Let us see where the singularities are. Solving for the roots of the discriminant we find 10 singularities at $u = 0$ which is formally $\psi = -\infty$ and two additional singularities at

$$u \sim -\frac{27i}{2}\sigma^3, \quad \frac{i}{64}\sigma^3 q^{-1} \quad (\text{B.9})$$

The first one is pushed to the boundary $\psi = 0$ in 5D while the second one becomes the single singularity at $\psi = 1$.

B.2. The general case

For the general case, with Wilson lines, we have to use the curve (A.17).

The τ of the corresponding torus can be calculated from the formula for the j -invariant of a curve in the form

$$\sum_{1 \leq i, j, k \leq 3} C_{ijk} X_i X_j X_k = 0. \quad (\text{B.10})$$

The result is an $SL(3, \mathbf{C})$ invariant rational function of the C_{ijk} 's and is given diagrammatically by:

$$\frac{\left(\begin{array}{c} \text{Diagram 1} \end{array} \right)^2}{\left(\begin{array}{c} \text{Diagram 2} \end{array} \right)^3} = \frac{4}{27} = \frac{256}{j(\text{curve})}$$

$\circ = \epsilon_{ijk}$
 $\bullet = C_{ijk}$

Fig.A: The j -function of the general cubic.

where a hollow circle is ϵ_{ijk} and a full circle is C_{ijk} and lines denote index contractions.

In (A.17) we use the ξ_i 's which are linear combinations of the w_i 's according to (A.13). The 5D Wilson lines are along the R_6 direction so we have

$$w_k = \frac{1}{2\pi i} \rho_k \log q \quad (\text{B.11})$$

and

$$\xi_k = \frac{1}{2\pi i} \gamma_k \log q \quad (\text{B.12})$$

where the γ_k are linear combinations of the real ρ_k according to (A.13). For small q and assuming $-\frac{1}{2} < \gamma_k < \frac{1}{2}$ we expand:

$$\begin{aligned} \frac{1}{(2\pi i)^2} \wp(\xi_k) &= \frac{1}{12} + \frac{q^{\gamma_k}}{(1 - q^{\gamma_k})^2} + \mathcal{O}(q^{1-\gamma_k}) \\ \frac{1}{(2\pi i)^3} \wp'(\xi_k) &= \frac{q^{\gamma_k}(1 + q^{\gamma_k})}{(1 - q^{\gamma_k})^3} + \mathcal{O}(q^{1-\gamma_k}). \end{aligned} \quad (\text{B.13})$$

The set of equations (A.16) becomes:

$$\sum_{n=0}^9 B_n q^{n\gamma_k} = 0, \quad k = 1 \dots 8 \quad (\text{B.14})$$

where the B_k -s are appropriate linear combinations of the Q_{ijk} -s.

We will assume:

$$|\gamma_1| > |\gamma_2| > \dots > |\gamma_8|.$$

We find the approximate solution:

$$B_0 = -1, \quad |B_k| = q^{-|\gamma_8| - |\gamma_7| - \dots - |\gamma_{8-k}| - k|\gamma_{9-k}|}, \quad k = 1 \dots 8. \quad (\text{B.15})$$

Substituting (B.15) into (A.17) and (Fig.A) we find that the dominant terms are

$$e^{2\pi i \tau} = q + \sum_{j=1}^9 c_j B_{j-1} B_8^{j-1} u^{-j} = \sum_{j=0}^9 c_j q^{k\psi - \alpha_k}. \quad (\text{B.16})$$

with

$$\begin{aligned} \alpha_0 &= -1, \quad \alpha_1 = 0, \\ \alpha_k &= 8(k-1)|\gamma_1| + (k-1) \sum_{2 \leq j \leq 9-k} |\gamma_j| + 2(k-1)|\gamma_{10-k}| + k \sum_{11-k \leq j \leq 8} |\gamma_j|, \quad k = 2 \dots 8 \\ \alpha_9 &= 9|\gamma_8| + 9|\gamma_7| + 9|\gamma_6| + 9|\gamma_5| + 9|\gamma_4| + 9|\gamma_3| + 9|\gamma_2| + 72|\gamma_1|. \end{aligned} \quad (\text{B.17})$$

Now we can read of the coupling constant from (B.16). It is (see Fig.B)

$$\frac{8\pi i}{g_5^2} = \left(\frac{1}{2\pi R_6} \right) \min_{k=0 \dots 9} (k\psi - \alpha_k), \quad \psi = \sqrt{2} R_6 \phi. \quad (\text{B.18})$$

This is a piecewise linear function.

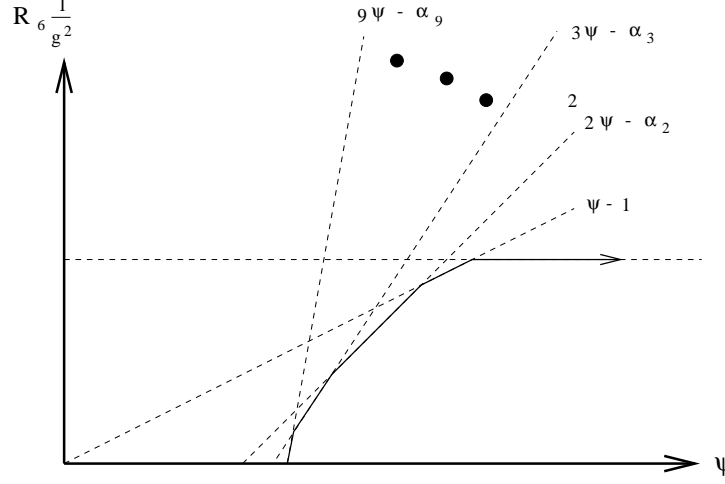


Fig.B: The 5D coupling constant is a piecewise linear function of ψ .

The number of discontinuities of the derivative could be anything between 1 and 9 according to the precise values of the α_k 's.

Formula (B.18) describes the general case for small Wilson lines. It includes the cases discussed in the previous subsections but it is not the *most* general case. When the γ_k 's are not necessarily all small, we should take account of more terms in the expansion of the \wp -functions in (B.13).

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